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ON THE "CONDITION AT THE EDGE" IN THE LINEAR THEORY OF LONG SURFACE WAVES

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The assumption that the kinetic energy contained in an arbitrary finite volume of fluid moving near the edge of a two-dimensional wedge is itself finite, is used as the basis for the investigation of asymptotic behavior of the form assumed by the surface of a perfect fluid near the edge. Explicit expressions are obtained for the form of the surface of a perfect fluid near the edge of a two-dimensional wedge, for basins of constant depth, at rest and rotating at a uniform rate.

Certain problems of the theory of wave motion in which nonsmooth boundaries (edges) appear, e. g. when the region in question is bounded by a nonsmooth surface, can have several mathematically correct solutions, but only one of these solutions will describe correctly the physical phenomenon investigated. In such a case additional physical constraints must be introduced in order to ensure the uniqueness. It is proposed that the requirement that the kinetic energy is finite in any finite volume of fluid moving near the edge be adopted as such a constraint, and be called the condition at the edge. The condition is equivalent to the requirement that

$$\int\limits_V \frac{\rho}{2} |v|^2 dV \to 0 \qquad (1)$$

when the volume V near the edge tends to zero. If the edge represents a smooth curve, then it can be replaced by a straight line at every of its points, and local cylindrical coordinates can be introduced at the edge. Condition (1) can be used to show that none of the components of the velocity vector v can increase faster than $r^{-1+\tau}$ $(\tau > 0)$. Strictly speaking, to obtain a unique solution of the wave equation it is sufficient to find the positive lower bound of the quantity τ. In many cases however is helpful if the exact value of τ is found.

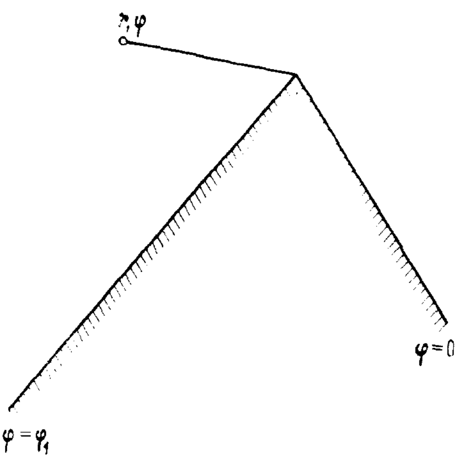

Fig. 1

We shall show how, using the linear theory of long surface waves we can derive the condition at the edge in a tank rotating uniformly with angular velocity σ about the z -axis directed along the edge of a perfectly rigid, two-dimensional wedge with an arbitrary opening angle (see Fig. 1). We assume that in the absence of rotation and perturbations, the free surface coincides with the plane $z = 0$ of the cylindrical r, φ, z -coordinate system. The velocity components v_r, v_φ and v_z as well as their derivatives with respect to r, φ and z, are assumed small, and the pressure dependence is determined by the hydrostatic relation

$$p = \rho g (\zeta - z) \tag{2}$$

where ρ is the fluid density, g is acceleration due to gravity and ζ is the rise in the fluid surface.

The Navier—Stokes equations have a known particular solution for these assumptions [1, 2] and the solution is

$$v_r = v_\varphi = v_z = 0, \quad \zeta = \frac{\sigma^2}{2g} r^2 + K(t) \tag{3}$$

We continue our analysis by introducing a new variable ζ_1, so that

$$\zeta_1 = \zeta - \left(\frac{\sigma^2}{2g} r^2 + K(t)\right) \tag{4}$$

Substituting (2) and (4) into the Navier—Stokes equations and the equation of continuity, we obtain the following system of equations for the three unknown functions v_r, v_φ and ζ_1:

$$\frac{\partial v_r}{\partial t} = -\frac{\partial \zeta_1}{\partial r} + 2\sigma v_\varphi \tag{5}$$

$$\frac{\partial v_\varphi}{\partial t} = -\frac{1}{r}\frac{\partial \zeta_1}{\partial r} - 2\sigma v_r$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{1}{h}\frac{\partial \zeta_1}{\partial t} = 0$$

We assume that the tank is of constant depth h, and the dependence of the unknown quantities on time is of the form $\exp i\omega t$.

Having demanded that the kinetic energy be finite in any finite volume of fluid moving near the edge, we find that v_r and v_φ vary near the edge as $r^{-1+\tau}$ $(\tau > 0)$ and can be written as $r \to 0$, in the form of power expansions

$$v_r = r^{-1+\tau}(a_0 + a_1 r + a_2 r^2 + \ldots) \tag{6}$$

$$v_\varphi = r^{-1+\tau}(b_0 + b_1 r + b_2 r^2 + \ldots)$$

We define the free surface in the same manner.

$$\zeta_1 = r^\tau (c_0 + c_1 r + c_2 r^2 + \ldots) \tag{7}$$

Substituting these expansions into Eqs. (5) and equating the coefficients of like power in r, we obtain the following system for the unknown functions in the zero approximation:

$$i\omega a_0 = -\tau c_0 + 2\sigma b_0, \quad i\omega b_0 = -\frac{\partial c_0}{\partial \varphi} - 2\sigma a_0, \quad \tau a_0 + \frac{\partial b_0}{\partial \varphi} = 0$$

This yields the equation for determining $b_0(\varphi)$

$$d^2 b_0 / d\varphi^2 + \tau^2 b_0 = 0$$

Solution $b_0 = A \sin(\tau\varphi) + B \cos(\tau\varphi)$ of the above equation determines the velocity v_φ normal to the wedge sides and vanishing at the sides. This yields $B = 0$, $A \sin \tau\varphi_1 = 0$. The problem has a nontrivial solution when

$$\sin \tau\varphi_1 = 0 \tag{8}$$

The upper bound of the order of singularity of the power series (6) and (7) is determined by the smallest root of Eqs. (8)

$$\tau = \pi / \varphi_1 \tag{9}$$

For example, near the edge of a half-plane standing on the bottom of the tank, the surface rise ζ_1 behaves as $r^{1/2}$ (in this case $\varphi_1 = 2\pi$).

Using the proposed scheme, we can find the order of singularity for the case when the tank is at rest. Equating the Coriolis and centrifugal forces to zero, we obtain an expression for τ which coincides exactly with the already known expression [(9).

In conclusion, it should be pointed out that the fact that the expression for τ is simple, is due to the character of the assumptions made. One of the assumptions was, that the wedge edge has infinite curvature. Adoption of this assumption leads to considerable reduction in the difficulties encountered during the actual tracing of the border line. In addition, combining this assumption with the condition at the edge of the form (9), yields satisfactory results in certain modelling problem of wave propagation in the geophysical basins. This was done in e. g. [3, 4].

For more accurate computations, a condition at the edge which takes into account the finite curvature of the edge, the length of the wave and possibly other parameters, becomes necessary.

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